NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS ADVANCED
YEAR 12 COURSE

Topic summary and exercises:

With references to

Ⓐ **Calculus with Exponential and Logarithmic Functions**
(For Ⓐ classes)

Name: ....

Initial version by H. Lam, October 2013.
Last updated March 26, 2025, with major revision in March 2020 for Mathematics Advanced.
Various corrections by students and members of the Mathematics Department at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at http://www.flaticon.com, used under CC BY 2.0.

Symbols used

⚠ Beware! Heed warning.

Ⓐ Mathematics Advanced content.

(X1) Mathematics Extension 1 content.

Literacy: note new word/phrase.

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MA12-6 applies appropriate differentiation methods to solve problems

MA12-7 applies the concepts and techniques of indefinite and definite integrals in the solution of problems

Syllabus subtopics

MA-C2 Differential Calculus

MA-C3 Applications of Differentiation

MA-C4 Integral Calculus

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 12 Advanced* or *CambridgeMATHS Year 12 Extension 1* will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Learning intentions & outcomes

Exponential Functions: Differentiation

✓ **Content/learning intentions**

☐ 22.1 Differentiation of exponential functions

☐ 22.2 Establish and use the formula $\frac{d}{dx}\left(a^x\right) = \left(\log_e a\right) a^x$

- Using graphing software or otherwise, sketch and explore the gradient function for a given exponential function, recognise it as another exponential function and hence determine the relationship between exponential functions and their derivatives

☐ 22.3 Apply the product, quotient and chain rules to differentiate functions of the form $f(x)g(x)$, $\frac{f(x)}{g(x)}$ and $f\Big(g(x)\Big)$ where $f(x)$ and $g(x)$ are any of the functions covered in the scope of this syllabus (ACMMM106) Also, establish using the chain rule: $\frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$ and use the exponential laws to simplify an expression before differentiating.

☐ 22.4 (B) Use any of the functions covered in the scope of this syllabus and their derivatives to solve practical and abstract problems

- Tangents and normals
- Optimisation
- Curve Sketching

Exponential Functions: Integration

✓ **Content/learning intentions**

☐ 22.5 Establish and use the formula $\int e^x \, dx = e^x + c$ and $\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + c$

☐ 22.6 (B) Calculate the area under a curve (ACMMM132)

✓ **Content/learning intentions**

☐ 22.7 (B) Calculate areas between curves determined by any functions within the scope of this syllabus (ACMMM134)

☐ 22.8 (B) Use the Trapezoidal rule to estimate areas under curves

Logarithmic Functions: Differentiation

✓ **Content/learning intentions**

☐ 22.9 Calculate the derivative of the natural logarithm function $\frac{d}{dx}(\ln x) = \frac{1}{x}$

☐ 22.10 Establish and use the formula $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

☐ 22.11 (B) Apply the product, quotient and chain rules to differentiate functions of the form $f(x)g(x)$, $\frac{f(x)}{g(x)}$ and $f\Big(g(x)\Big)$ where $f(x)$ and $g(x)$ are any of the functions covered in the scope of this syllabus (ACMMM106) Also, establish using the chain rule: $\frac{d}{dx}\Big(\ln f(x)\Big) = \frac{f'(x)}{f(x)}$ and use the logarithmic laws to simplify an expression before differentiating.

☐ 22.12 (B) Use any of the functions covered in the scope of this syllabus and their derivatives to solve practical and abstract problems

- Tangents and normals
- Curve Sketching
- Optimisation

Logarithmic Functions: Integration

✓ **Content/learning intentions**

☐ 22.13 Establish and use the formula $\int \frac{1}{x}\,dx = \ln|x| + c$ and $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$ for $x \neq 0,\ f(x) \neq 0$ respectively

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☐ 22.14 (B) Establish and use the formulae $\int a^x\,dx = \dfrac{a^x}{\ln a} + c$

- Use the change of base rule

☐ 22.15 (B) Calculate the area under a curve (ACMMM132)

☐ 22.16 (B) Calculate areas between curves determined by any functions within the scope of this syllabus (ACMMM134)

- Functions whose primitive is a logarithmic function in this case, or if via a transformation to an alternative axis, the exponential function.

☐ 22.17 (B) Use the Trapezoidal rule to estimate areas under curves

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Curve Sketching with trigonometric functions

✓ Content/learning intentions

☐ 23.1 Examine and apply transformations to sketch functions of the form $y = kf\Big(a(x+b)\Big) + c$, where a, b, c and k are constants, in a variety of contexts, where $f(x)$ is one of $\sin x$, $\cos x$ or $\tan x$, stating the domain and range when appropriate.

- Use technology or otherwise to examine the effect on the graphs of changing
  - the amplitude (where appropriate) $y = kf(x)$
  - the period $y = f(ax)$
  - the phase $f(x+b)$
  - the vertical shift $y = f(x) + c$
- Use k, a, b and c to describe transformational shifts and sketch graphs.

☐ 23.2 Solve trigonometric equations involving functions of the form $kf\Big(a(x+b)\Big) + c$, using technology or otherwise, within a specified domain

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☐ 23.3 Use trigonometric functions of the form $kf\Big(a(x+b)\Big) + c$ to model and/or solve practical problems involving periodic phenomena

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Contents

Part I

Calculus with Exponential Functions

Section 1

Differentiation involving the exponential function

Learning Goal(s)

Knowledge
Transforming and differentiating the exponential function

Skills
Differentiating the exponential function

Understanding
When to apply other differentiation techniques

By the end of this section am I able to:

22.1 Differentiation of exponential functions

22.2 Establish and use the formula $\frac{d}{dx}(a^x) = (\log_e a)\, a^x$ Apply the product, quotient and chain rules to differentiate functions of the form $f(x)g(x)$, $\frac{f(x)}{g(x)}$ and $f\big(g(x)\big)$ where $f(x)$ and $g(x)$ are any of the functions covered in the scope of this syllabus

22.3 Use any of the functions covered in the scope of this syllabus and their derivatives to solve practical and abstract problems

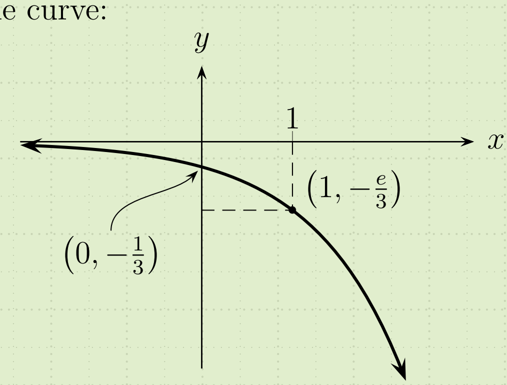
1.1 Curve sketching without calculus

Important note

- This section reviews work in **Year 11 Topic 8 - Exponentials and Logarithms**. Only do as much as you have to.
- Consequently, only harder examples are given.

Example 1

[**Ex 5A/6A Q9**] The graph is a dilation of $y = e^x$. Describe the dilation, and write down the equation of the curve:



Timed exam practice 1 (Allow approximately 1 minute)

[2023 NBHS Adv Task 3] (1 mark) The function $f(x) = e^{x+2}$ is transformed into the function $g(x) = e^{x+4}$. Which of the following describes the effect of this transformation?

(A) A horizontal dilation

(B) A vertical dilation

(C) A vertical translation

(D) A reflection about the y axis

Further exercises

(A) Ex 5A **(X1) Ex 6A**

- Only as much as you have to, to enjoy fluency in curve sketching

1.2 Differentiation

Example 2

[2017 2U HSC Q3] What is the derivative of e^{x^2}?

(A) $x^2e^{x^2-1}$

(B) $2xe^{2x}$

(C) $2xe^{x^2}$

(D) $2e^{x^2}$

Example 3

[2018 2U HSC Q11] (2 marks) Differentiate $\dfrac{e^x}{x+1}$.

Example 4

[Ex 5B/6B Q18] Define two functions $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

(a) Show that $\dfrac{d}{dx}(\cosh x) = \sinh x$ and $\dfrac{d}{dx}(\sinh x) = \cosh x$.

(b) Find the second derivative of each function, and show that they both satisfy $y'' = y$.

(c) Show that $\cosh^2 x - \sinh^2 x = 1$.

Further exercises

Ⓐ **Ex 5B** (X1) **Ex 6B**

- Only as much as you have to, to enjoy fluency in differentiating via product/quotient/chain rules.

1.2.1 Change of base

Theorem 1

$$\frac{d}{dx}\left(a^x\right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Proof

Steps

1. Let $y = a^x$. Take logarithm base e on both sides:

$$\therefore \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots$$

2. Exponentiate both sides, to base e:

$$y = \ldots\ldots\ldots\ldots\ldots\ldots$$

3. Differentiate, noting the constants in the expression:

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

4. Replace $e^{x(\log_e a)}$ with the expression it replaced:

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots$$

GeoGebra

Check that the derivative of $y = a^x$ is simply a constant multiple of its corresponding y value.

1.3 Applications of differentiation

1.3.1 Tangents

Example 5

[**2014 2U HSC Q15**] The line $y = mx$ is a tangent to the curve $y = e^{2x}$ at the point P.

i. Sketch the line and the curve on one diagram. **1**

ii. Find the coordinates of P. **3**

iii. Find the value of m. **1**

Answer: i. Verify with technology. ii. $P\left(\frac{1}{2}, e\right)$ iii. $m = 2e$

1.3.2 Curve sketching

Theorem 2

The function e^x .... over any algebraic power x^k, $k \in \mathbb{Z}^+$ as $x \to \pm\infty$

$$\lim_{x\to\infty} x^k e^{-x} = \ldots \qquad \lim_{x\to-\infty} x^k e^{x} = \ldots$$

Example 6

[2024 SGS Adv Trial Q29] Let $f(x) = xe^{-x^2}$.

(a) Find the coordinates of the stationary points of $y = f(x)$. **3**
You do *not* need to check the nature of the stationary points.

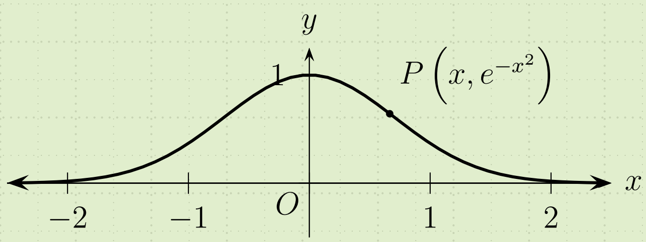
Answer: $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2e}}\right)$

(b) Without using any further calculus, sketch the graph of $y = f(x)$, showing stationary points, any intercepts and asymptotes. **2**

1.3.3 Optimisation

Example 7

[2018 Independent 2U Q16] In the diagram, $P\left(x, e^{-x^2}\right)$ is a variable point on the curve $y = e^{-x^2}$.



i. Show that the function $f(x) = e^{-x^2}$ is an even function. **1**

ii. Show that $OP^2 = x^2 + e^{-2x^2}$. **1**

iii. Find in simplest exact form, the minimum value of OP^2 as the point P moves on the curve. **3**

Answer: $\frac{1}{2}(\ln 2 + 1)$

1.3.4 Modelling

Example 8

[2020 Adv HSC Q21] Hot tea is poured into a cup. The temperature of tea can be modelled by $T = 25 + 70(1.5)^{-0.4t}$, where T is the temperature of the tea, in degrees Celsius, t minutes after it is poured.

(a) What is the temperature of the tea 4 minutes after it has been poured? **1**

(b) At what rate is the tea cooling 4 minutes after it has been poured? **2**

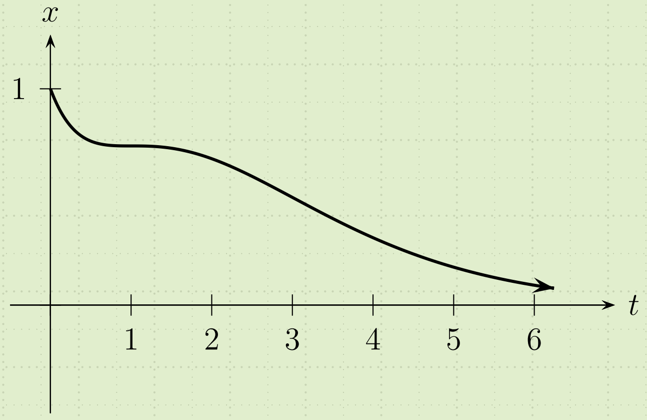
(c) How long after the tea is poured will it take for its temperature to reach 55°C? **3**

Answer: (a) 61.6°C (b) 5.9°C per minute (c) 5.22 minutes

1.3.5 Motion

Example 9

[**2023 Independent Adv Trial Q29**] (5 marks) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line given by $x = (t^2 + 1)\, e^{-t}$ and velocity v ms^{-1}. The diagram shows the graph of x as a function of t.



(a) Express v as a function of t and hence find when the particle is at rest. **2**

(b) Use the features of the graph to **explain** whether the velocity and acceleration of the particle are positive or negative over the period of time when it is speeding up. **2**

(c) **Describe** the limiting behaviour (speed and position) of the particle. **1**

Important note

Describe and explain questions require brief sentences!

1.3.6 Additional questions

1. **[1998 2U HSC Q6]** The function $f(x) = xe^{-2x} + 1$ has first derivative $f'(x) = e^{-2x} - 2xe^{-2x}$ and second derivative $f''(x) = 4xe^{-2x} - 4e^{-2x}$.

i. Find the value of x for which $y = f(x)$ has a stationary point. **1**

ii. Find the values of x for which $f(x)$ is increasing. **1**

iii. Find the value of x for which $y = f(x)$ has a point of inflection and determine where the graph of $y = f(x)$ is concave upwards. **2**

iv. Sketch the curve $y = f(x)$ for $-\frac{1}{2} \leq x \leq 4$. **2**

v. Describe the behaviour of the graph for very large values of x. **1**

Answers

1. (a) $x = \frac{1}{2}$ (b) $x < \frac{1}{2}$ (c) $x = 1$ (d) Sketch (e) As $x \to \infty$, $xe^{-2x} \to 0^+$. Hence $1 + xe^{-2x} \to 1^+$. Also, as $x \to -\infty$, $xe^{-2x} \to -\infty$. Hence $1 + xe^{-2x} \to -\infty$.

Further exercises

(A) **Ex 5C** (X1) **Ex 6C**

- Only as much as you have to, to review applications of differentiation involving the exponential function.

(A) **Ex 7B**

- Q6, 8, 16

(A) **Ex 7D**

- Q6, 8, 14

(X1) **Ex 9B**

- Q5, 14

(X1) **Ex 9D**

- Q6, 11, ⚠ 13

Section 2

Integration involving the exponential function

Learning Goal(s)

Knowledge
Various integration techniques involving the exponential function

Skills
Finding primitives resulting in $e^{f(x)}$

Understanding
Why $\int f'(x)e^{f(x)}\,dx = e^{f(x)} + C$

By the end of this section am I able to:

22.4 Establish and use the formula $\displaystyle\int e^x\,dx = e^x + c$ and $\displaystyle\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$

22.5 Calculate the area under a curve

22.6 Calculate areas between curves determined by any functions within the scope of this syllabus

22.7 Use the Trapezoidal rule to estimate areas under curves

2.1 Primitives resulting in e^x

- Given $\dfrac{d}{dx}(e^x) = e^x$, then $\displaystyle\int e^x\,dx = e^x + C$

Laws/Results

- By the chain rule, $\dfrac{d}{dx}\left(e^{f(x)}\right) =$ ....
- Hence, $\displaystyle\int$ .... $dx = e^{f(x)} + C$.

Important note

⚠ Think of "what needs to be differentiated to obtain the integrand?".

⚠ .... will most likely be useful.

2.1.1 Indefinite integrals

Example 10

Evaluate:

1. $\displaystyle\int e^{-3x+2}\,dx.$

2. $\displaystyle\int \left(e^x + e^{-3x}\right)\,dx$

3. $\displaystyle\int (e^x+3)(e^x-4)\,dx$

4. $\displaystyle\int \frac{e^{2x}+1}{e^x}\,dx$

5. $\displaystyle\int \left(e^{3x}-2\right)^2\,dx$

6. $\displaystyle\int 6^x\,dx$

Answers

1. $-\frac{4}{3}e^{-3x+2}+C$ **2.** $e^x-\frac{1}{3}e^{-3x}$ **3.** $\frac{1}{2}e^{2x}-e^x-12x+C$ **4.** $e^x-e^{-x}+C$ **5.** $\frac{1}{6}e^{6x}-\frac{4}{3}e^{3x}+4x+C$ **6.** $\frac{1}{\ln 6}6^x+C$

2.1.2 Definite integrals

Example 11

Evaluate $\displaystyle\int_2^3 e^{2-x}\,dx$.

Answer: $\frac{e-1}{e}$

Example 12

Evaluate $\displaystyle\int_0^3 \left(2+e^{-x}\right)\,dx$.

Answer: $7-\frac{1}{e^3}$

Example 13

Evaluate $\displaystyle\int_0^2 \left(e^{2x}+3\right)^2\,dx$.

Answer: $\frac{1}{4}\left(59+12e^4+e^8\right)$

2.1.3 Harder primitives

Example 14

Evaluate $\displaystyle\int x^2 e^{x^3}\,dx$

Answer: $\frac{1}{3}e^{x^3}+C$

Example 15

Evaluate $\displaystyle\int (3x-6)e^{x^2-4x-3}\,dx$

Answer: $\frac{3}{2}e^{x^2-4x-3}+C$

Example 16

Evaluate $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

Answer: $2e^{\sqrt{x}}+C$

Further exercises

(A) **Ex 5D**

- Q1-4 last 2 columns
- Q7-18

(X1) **Ex 6D**

- Q1 last column
- Q2 last 2 columns
- Q4
- Q5 last column
- Q7-9 last 2 columns
- Q11-18

2.2 Applications of integration

2.2.1 Transformations

Theorem 3

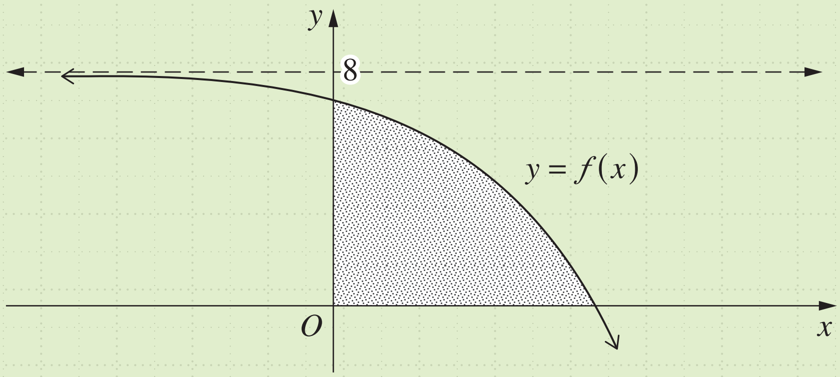
Integrands with a base other than e

$$\int a^x\,dx = \text{..........................}$$

$$\int f'(x)a^{f(x)}\,dx = \text{.................................}$$

Example 17

[2021 Adv HSC Q28] The region bounded by the graph of the function $f(x) = 8 - 2^{-x}$ and the coordinate axes is shown.



(a) Show that the exact area of the shaded region is given by $24 - \dfrac{7}{\ln 2}$ **3**

(b) A new function $g(x)$ is found by taking the graph of $y = -f(-x)$ and translating it by 5 units to the right. **2**

Sketch the graph of $y = g(x)$ showing the x intercept and the asymptote.

(c) Hence, find the exact value of $\displaystyle\int_2^5 g(x)\,dx$ **1**

2.2.2 Simple areas

Important note

- .... , if none is available!
- Check for x axis crossings.

Example 18

Find the area between the curve $y = e^x - e$ and the coordinate axes. **Answer:** 1

Example 19

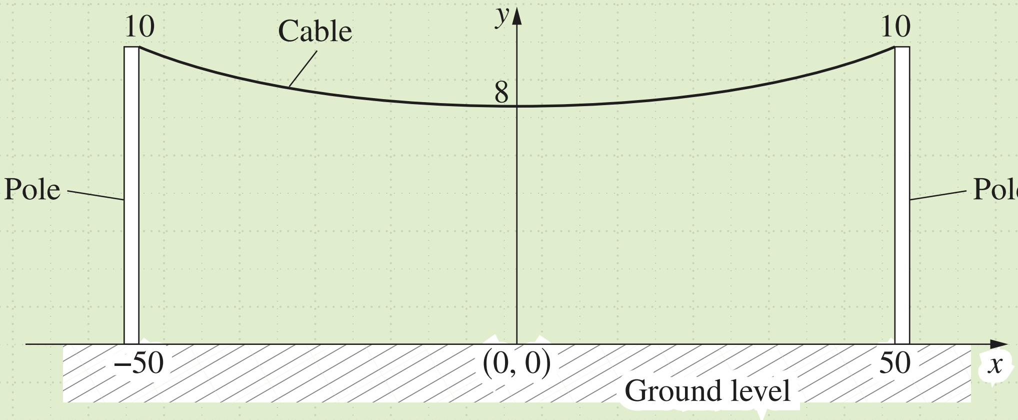
[**2013 CSSA Trial**] Using the Trapezoidal Rule, find an approximation for

$$\int_0^2 e^{x^2}\, dx$$

with 3 function values. **Answer:** $\frac{1}{2}\left(1 + 2e + e^4\right)$

Example 20

[2020 Adv HSC Sample Q38] A cable is freely suspended between two 10 m poles, as shown. The poles are 100 metres apart and the minimum height of the cable is 8 metres.



The height of the cable is given by $y = c\left(e^{kx} + e^{-kx}\right)$ where c and k are positive constants.

(a) Show that the value of c is 4. **1**

(b) Use the result in part (a) to show that one value of k is $\dfrac{\ln 2}{50}$ **4**

(c) Hence find the area between the poles, the cable and the ground. **3**

Answer: $\frac{600}{\ln 2} \approx 865\,\mathrm{m}^2$

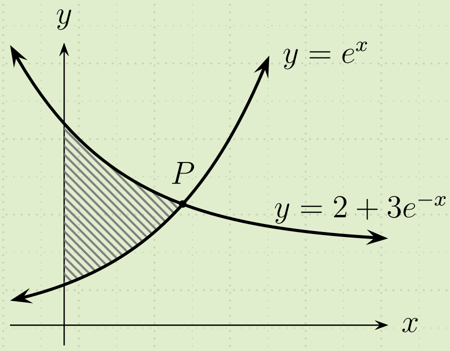
2.2.3 Area between two curves

Important note

- .... , if none is available!
- Check for curves crossing over each other.

Example 21

[2011 NSGHS 2U Trial] The diagram shows the graphs of $y = e^x$ and $y = 2 + 3e^{-x}$ intersecting at the point P.



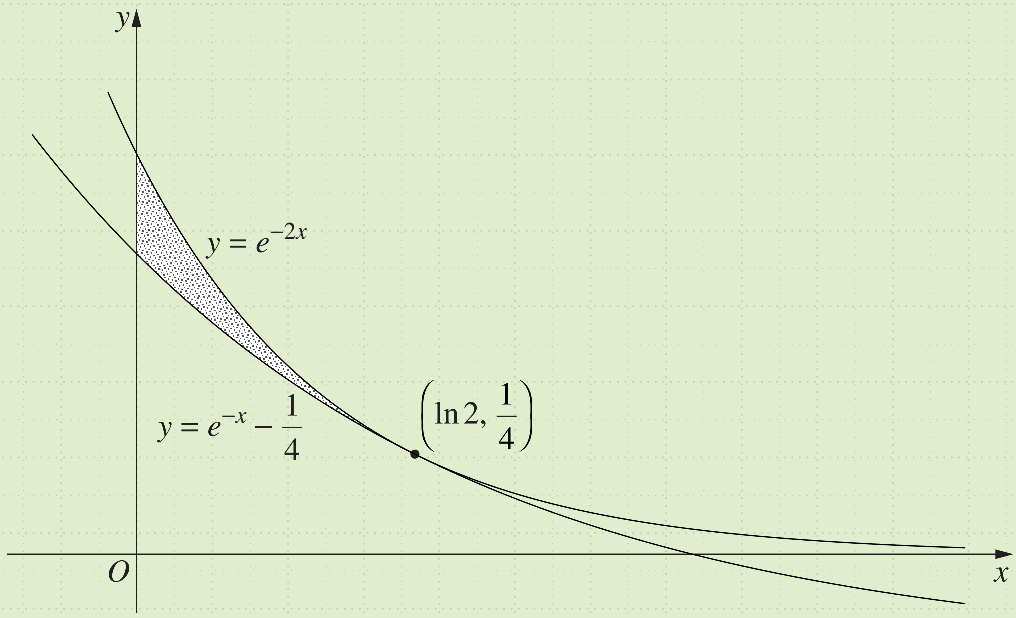
(i) Show that the curves intersect when $e^{2x} - 2e^x - 3 = 0$ **1**

(ii) Hence show that the x coordinate of the point P is $\log_e 3$. **2**

(iii) Hence find the exact area of the shaded region. **3**

Example 22

[2023 Adv HSC Q32] The curves $y = e^{-2x}$ and $y = e^{-x} - \dfrac{1}{4}$ intersect at exactly one point as shown in the diagram. The point of intersection has coordinates $\left(\ln 2, \dfrac{1}{4}\right)$. (Do NOT prove this.)



(a) Show that the area bounded by the two curves and the y axis, as shaded in the diagram, is **3**

$$\frac{1}{4}\ln 2 - \frac{1}{8}$$

(b) Find the values of k such that the curves $y = e^{-2x}$ and $y = e^{-x} + k$ intersect at two points. **3**

Marking criteria

(a) ✓ [1] Provides an integral expression for the area, or equivalent merit

✓ [2] Provides an antiderivative, or equivalent merit

✓ [3] Provides correct solution

(b) ✓ [1] Attempts to form a quadratic equation, or equivalent merit

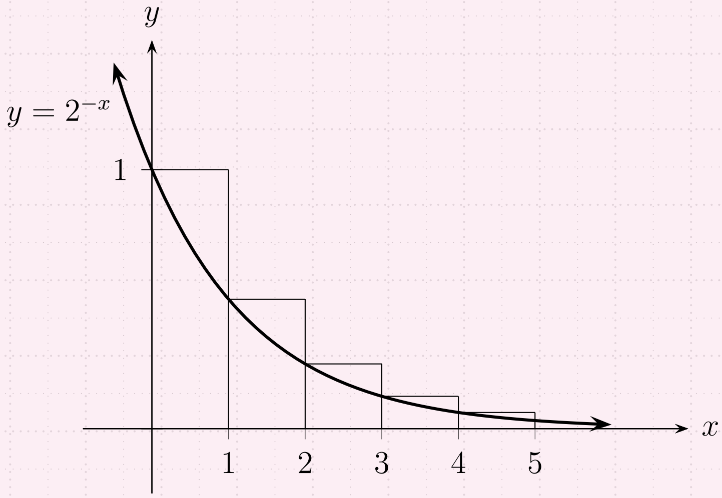
✓ [2] Uses the discriminant to find $k > -\frac{1}{4}$, or equivalent merit

✓ [3] Provides correct solution $-\frac{1}{4} < k < 0$

Timed exam practice 2 (Allow approximately 8 minutes)

[2022 Adv HSC Q29]

(a) The diagram shows the graph of $y = 2^{-x}$. Also shown on the diagram are the first 5 of an infinite number of rectangular strips of width 1 unit and height $y = 2^{-x}$ for non-negative integer values of x. For example, the second rectangle shown has width 1 and height $\frac{1}{2}$. **1**



The sum of the areas of the rectangles forms a geometric series.

Show that the limiting sum of this series is 2.

(b) Show that $\displaystyle\int_0^4 2^{-x}\,dx = \frac{15}{16\ln 2}$. **2**

(c) Use parts (a) and (b) to show that $e^{15} < 2^{32}$. **2**

Marking criteria

(a) ✓ [1] Shows the limiting sum

(b) ✓ [1] Substitutes correctly into anti -derivative, or equivalent merit

✓ [2] Provides correct solution

(c) ✓ [1] Sets up correct inequality, or equivalent merit

✓ [2] Provides correct solution

2.2.4 Rates of change

Timed exam practice 3 (Allow approximately 7 minutes)

[2021 Adv HSC Q23] (4 marks) A population, P, which is initially 5 000, varies according to the formula

$$P = 5\,000 b^{-\frac{t}{10}}$$

where b is a positive constant and t is time in years, $t \geq 0$.

The population is 1 250 after 20 years.

Find the value of t, correct to one decimal place, for which the instantaneous rate of decrease is 30 people per year. **Answer:** $t = 35.3$

Marking criteria

- ✓ [1] Provides correct derivative, or equivalent merit
- ✓ [2] Finds b and provides correct derivative, or equivalent merit
- ✓ [3] Equates -30 to a correct derivative, with $b = 2$, or equivalent merit
- ✓ [4] Provides correct solution

2.2.5 Motion

Example 23

[2017 2U HSC Q15] Two particles move along the x axis.

When $t = 0$, particle P_1 is at the origin and moving with velocity 3.

For $t \geq 0$, particle P_1 has acceleration given by $a_1 = 6t + e^{-t}$.

(i) Show that the velocity of particle P_1 is given by $v_1 = 3t^2 + 4 - e^{-t}$. **2**

When $t = 0$, particle P_2 is also at the origin.

For $t \geq 0$, particle P_2 has velocity given by $v_2 = 6t + 1 - e^{-t}$.

(ii) When do the two particles have the same velocity? **2**

(iii) Show that the particles do not meet for $t > 0$. **3**

Timed exam practice 4 (Allow approximately 3 minutes)

[2019 2U HSC Q14] (2 marks) A particle is moving along a straight line. The particle is initially at rest. The acceleration of the particle at time t seconds is given by $a = e^{2t} - 4$, where $t \geq 0$.

Find an expression, in terms of t, for the velocity of the particle.

Answer: $v = \frac{1}{2}e^{2t} - 4t - \frac{1}{2}$

Marking criteria

✓ [1] Attempts to integrate, or equivalent merit

✓ [2] Provides correct solution

Further exercises

Ⓐ **Ex 5E**

- All questions, except Q1-2

Ⓐ **Ex 7B**

- Q8, 16

Ⓐ **Ex 7C**

- Q8, 9: parts (c), (d)
- Q16

Ⓐ **Ex 7E**

- Q1(e), 8, 9

(X1) **Ex 6E**

- Q2-17

(X1) **Ex 9B**

- Q5, 14

(X1) **Ex 9C**

- Q4(a), (b): part ii
- Q12

(X1) **Ex 9F**

- Q5, 6

2.2.6 Further questions

1. **[Legacy Ex 13C Q13]**

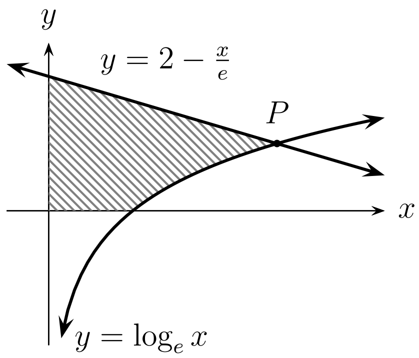
(a) Differentiate $y = x^2 e^{-x^2}$.

(b) Hence show that

$$\int x^3 e^{-x^2}\,dx = -\frac{e^{-x^2}}{2}\left(x^2+1\right) + C$$

and calculate $\int_1^2 x^3 e^{-x^2}\,dx$.

2. **[2011 Independent 2U Trial]** (X1) The shaded region below represents the area bounded by the x and y axes and the curves $y = 2 - \frac{x}{e}$ and $y = \log_e x$.



(i) Show by substitution that the curves $y = 2 - \frac{x}{e}$ and $y = \log_e x$ intersect at the point $P(e, 1)$. **1**

(ii) Hence find the exact area of the shaded region. **Answer:** $\frac{3e}{2} - 1$ **4**

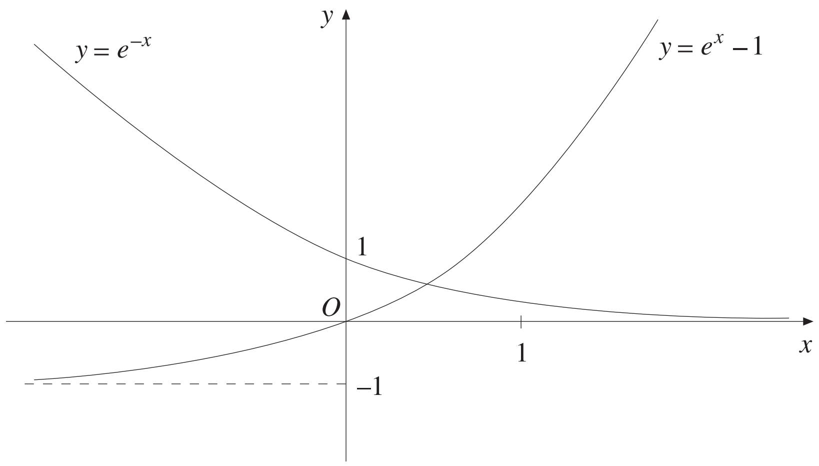
3. **[2020 Adv HSC Sample Q30]** (3 marks) The population, P, of rabbits on an island is given by $P(t)$, where t is the time in years after the rabbits were introduced. The rabbit population changes at a rate modelled by the function $\frac{dP}{dt} = 30e^{1.25t}$.

Calculate the increase in the number of rabbits at the end of the first 10 years. Give your answer correct to two significant figures. **Answer:** 6 400 000

4. [1996 2U HSC Q5]

(b) Solve the equation $u^2 - u - 1 = 0$, correct to three decimal places. **2**

(c) The diagram shows the graphs of $y = e^x - 1$ and $y = e^{-x}$.



i. Find the area between the curves from $x = 1$ to $x = 2$. Leave your answer in terms of e. **3**

ii. Show that the curves intersect when **2**

$$e^{2x} - e^x - 1 = 0$$

iii. Use the results of part (b) with $u = e^x$ to show that the x coordinate of the point of intersection of the curves is approximately 0·481. **1**

Part II

Calculus with Logarithmic Functions

Section 3

Differentiation involving the logarithmic function

Learning Goal(s)

Knowledge
Transforming and differentiating the logarithmic function

Skills
Differentiating the logarithmic function

Understanding
Why $\frac{d}{dx}(\log_e x) = \frac{1}{x}$, and when to apply other differentiation techniques

By the end of this section am I able to:

22.9 Calculate the derivative of the natural logarithm function $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

22.10 Establish and use the formula $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$

22.11 Apply the product, quotient and chain rules to differentiate functions of the form $f(x)g(x)$, $\dfrac{f(x)}{g(x)}$ and $f\big(g(x)\big)$ where $f(x)$ and $g(x)$ are any of the functions covered in the scope of this syllabus

22.12 Use any of the functions covered in the scope of this syllabus and their derivatives to solve practical and abstract problems

3.1 Curve sketching without calculus and equations

Example 24

Solve $3e^{2x} - 11e^x - 4 = 0$. **Answer:** $\ln 4$

Example 25

Solve $\ln x - \dfrac{9}{\ln x} = 0$ **Answer:** e^3 or e^{-3}

Example 26

Sketch $y = \log_e(5 - x)$ by transforming the graph of $y = \log_e x$.

Timed exam practice 5 (Allow approximately 1 minute)

[2024 NBHS Adv Task 3] The function $f(x) = \log_e x$ is transformed to $g(x) = \log_{10}(2x)$ by using only two function transformations.

Which of the following transformations, applied in the order given, will transform $f(x)$ into $g(x)$?

(A)
1. Apply a vertical dilation factor of $\log_{10} e$
2. Apply a horizontal dilation factor of 2

(B)
1. Apply a vertical dilation factor of $\dfrac{1}{\log_{10} e}$
2. Apply a horizontal dilation factor of 2

(C)
1. Apply a vertical dilation factor of $\log_{10} e$
2. Apply a vertical translation of $(\log_{10} 2)$ units in the upwards direction.

(D)
1. Apply a vertical dilation factor of $\dfrac{1}{\log_{10} e}$
2. Apply a vertical translation of $(\log_{10} 2)$ units in the upwards direction.

3.1.1 Additional questions

1. [**Ex 6F Q17**] Use and describe:

(a) a dilation of $y = \log_e x$, to sketch $y = \log_e 2x$.

(b) a subsequent translation to sketch $y = \log_e 2(x-1)$.

(c) a subsequent dilation to sketch $y = \frac{1}{2}\log_e 2(x-1)$.

(d) a subsequent translation to sketch $y = \frac{1}{2}\log_e 2(x-1) - 2$.

Further exercises

Ⓐ **Ex 5F**

- Q2-16

(X1) **Ex 6F**

- Q5-16

Only as many as required to be fluent at equation solving/curve sketching.

3.2 Differentiation

3.2.1 Derivation of result

Steps

1. If $y = \log_e x$.

2. Change subject to x: ....

3. Differentiate x with respect to y: ....

4. Use the fact that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$:

Laws/Results

If $y = \log_e x$ (or $y = \ln x$), the derivative is

....

Important note

The .... is almost always required.

3.2.2 Standard questions

Example 27

Evaluate:

1. $\frac{d}{dx}\left(x + \log_e x\right)$ **2.** $\frac{d}{dx}\left(\log_e\left(x^2+1\right)\right)$ **3.** $\frac{d}{dx}\Big(\log_e\left(\log_e x\right)\Big)$

Answers

1. $1 + \frac{1}{x}$ **2.** $\frac{2x}{x^2+1}$ **3.** $\frac{1}{x\log_e x}$

Laws/Results

Shortcut: If $y = \log_e\Big(f(x)\Big)$ or $y = \ln\Big(f(x)\Big)$, the derivative is

....

where $f'(x)$ is the ....

3.2.3 Miscellaneous questions

Example 28

Differentiate $y = (x + \ln x)^4$

Answer: $4\left(1 + \frac{1}{x}\right)(x + \ln x)^3$

Example 29

Differentiate $y = \dfrac{\ln(3x+1)}{2x+3}$.

Answer: $\frac{3(2x+3)-2(3x+1)\ln(3x+1)}{(2x+3)^2(3x+1)}$

3.2.4 Simplification via log laws

Example 30

Evaluate $\dfrac{d}{dx}\left(\log_e\left(\dfrac{1+x^2}{1-x^2}\right)\right)$

Answer: $2x\left(\frac{1}{1+x^2}+\frac{1}{1-x^2}\right)$

Example 31

Evaluate $\dfrac{d}{dx}\left(\log_e\sqrt{\dfrac{3x+4}{5x^2+1}}\right)$

Answer: $\frac{1}{2}\left(\frac{3}{3x+4}-\frac{10x}{5x^2+1}\right)$

Example 32

Differentiate $y=\log_e\left[\left(x^3+3\right)\left(x^2+3x+1\right)\right]$.

Answer: $\frac{3x^2}{x^3+3}+\frac{2x+3}{x^2+3x+1}$

3.2.5 Change of base

Theorem 4

$$\frac{d}{dx}\left(\log_a x\right) = \ldots\ldots\ldots\ldots$$

Proof

Steps

1. Let $y = \log_a x$. Exponentiate both sides, to base a:

$$\therefore \ldots\ldots = \ldots$$

2. Take logarithm base e on both sides:

$$\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$$

3. Use a logarithm law to simply the y expression, and then divide:

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

4. Differentiate, noting the constants in the expression:

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

5. Fully simplify:

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots$$

Example 33

Evaluate $\dfrac{d}{dx}\Big(\log_3(2x^3+3x)\Big)$

Answer: $\frac{1}{\ln 3}\left(\frac{6x^2+3}{2x^3+3x}\right)$

Example 34

[2020 Adv HSC Sample Q10] Given the function $y = \log_7\left(x^x\right)$, which expression is equal to $\dfrac{dy}{dx}$?

(A) $\dfrac{1}{x\ln 7}$

(B) $\dfrac{1}{\ln 7} \times \log_7\left(x^{x-1}\right)$

(C) $\dfrac{1}{x^2\ln 7}$

(D) $\log_7 x + \dfrac{1}{\ln 7}$

Further exercises

Ⓐ **Ex 5G**

- Q1-4 last 2 columns
- Q5-14, 17-18

(x1) **Ex 6G**

- Q3-4
- Q5-12, 14 last 2 columns
- Q13

3.2.6 Supplementary Exercises

Differentiate the following with respect to x.

1. $y = 4\log_e x$

2. $y = \dfrac{\log_e x}{14}$

3. $y = \log_e(3x-4)$

4. $y = \log_e(5x-6)$

5. $y = \log_e\left(x^2\right)$

6. $y = \log_e\left(ax^2+b\right)$

7. $y = \log_e\left(1-x^2\right)$

8. $y = \log_e\left(-x^2+6x\right)$

9. $y = \log_e\left(2x^3+3x\right)$

10. $y = \log_e\left(2x^4-3x^2+2x+1\right)$

11. $y = x\log_e x$

12. $y = x^2\log_e x$

13. $y = x^2\log_e(4x+3)$

14. $y = (2x+5)^2\log_e(2x+5)$

15. $y = \log_e\left(x^2+x\right)$

16. $y = \dfrac{\log_e x}{x}$

17. $y = \dfrac{x^2}{\log_e x}$

18. $y = \dfrac{x^2-1}{\log_e x}$

19. $y = \log_e x^{100}$

20. $y = \log_e\left(x^2+4x+5\right)^3$

21. $y = 6\log_e\sqrt[3]{x}$

22. $y = \log_e\sqrt{1+x^2}$

23. $y = \log_e\left(\dfrac{x^2}{1+x^2}\right)$

24. $y = \log_e\left(\dfrac{1+x}{1-x}\right)$

25. $y = \log_e\left(\dfrac{2x+3}{3x-4}\right)$

26. $y = \log_e\sqrt[4]{\dfrac{1+x^2}{1-x^2}}$

27. $y = \log_e\sqrt{\dfrac{x^4-1}{x^4+1}}$

28. $y = \log_e\left[\left(x^2+2\right)^2\left(x^3+x-1\right)\right]$

29. $y = \log_e\left[(4x+2)^4(8x-3)^6\right]$

30. $y = \log_e\left(x\sqrt{2x+1}\right)$

31. $y = \log_e\dfrac{x}{\sqrt{2x+1}}$

32. $y = \left(x^2+1\right)\log_e(2x+1)$

33. $y = (ax+b)\log_e(ax)$

34. $y = \log_e x^3 + \log_e^3 x$

35. $y = x^{\log_e 3}$

36. $y = {\log_e}^4(ax)$

37. $y = {\log_e}^2(2x+3)$

38. $y = x\log_e\sqrt{x-1}$

39. $y = \log_e\left(x^2\sqrt{3x-2}\right)$

40. $y = \sqrt{4+\log_e x}$

41. $y = \log_e\left(x+\sqrt{1+x^2}\right)$

Answers

1. $\frac{4}{x}$ **2.** $\frac{1}{14x}$ **3.** $\frac{3}{3x-4}$ **4.** $\frac{5}{5x-6}$ **5.** $\frac{2}{x}$ **6.** $\frac{2ax}{ax^2+b}$ **7.** $\frac{2x}{x^2-1}$ **8.** $\frac{-2x}{x^2+6x}$ **9.** $\frac{6x^2+3}{2x^3+3x}$ **10.** $\frac{8x^3-6x+2}{2x^4-3x^2+2x+1}$ **11.** $1+\log_e x$ **12.** $x+2x\log_e x$ **13.** $\frac{4x^2}{4x+3}+2x\log_e(4x+3)$ **14.** $2(2x+5)\left(1+2\ln(2x+5)\right)$ **15.** $\frac{2x+1}{x^2+x}$ **16.** $\frac{1-\log_e x}{x^2}$ **17.** $\frac{x(2\log_e x-1)}{(\log_e x)^2}$ **18.** $\frac{2x^2\log_e x-x^2+x}{x(\log_e x)^2}$ **19.** $\frac{100}{x}$ **20.** $\frac{3(2x+4)}{x^2+4x+5}$ **21.** $\frac{2}{x}$ **22.** $\frac{x}{x^2+1}$ **23.** $\frac{2}{x}-\frac{2x}{1+x^2}$ **24.** $\frac{1}{1+x}+\frac{1}{1-x}$ **25.** $\frac{2}{2x+3}-\frac{3}{3x-4}$ **26.** $\frac{1}{2}\left(\frac{x}{1+x^2}+\frac{x}{1-x^2}\right)$ **27.** $2\left(\frac{x^3}{x^4-1}-\frac{x^3}{x^4+1}\right)$ **28.** $\frac{4x}{x^2+2}+\frac{3x^2+1}{x^3+x-1}$ **29.** $\frac{8}{2x+1}+\frac{48}{8x-3}$ **30.** $\frac{3x+1}{2x^2+x}$ **31.** $\frac{1}{x}-\frac{1}{2x+1}$ **32.** $\frac{2\left(x^2+1\right)}{2x+1}+2x\log_e(2x+1)$ **33.** $\frac{1}{x}(ax+b)+a\log_e(ax)$ **34.** $\frac{3}{x}\left(1+(\log_e x)^2\right)$ **35.** $(\log_e 3)x^{\log_e 3-1}$ **36.** $\frac{4a}{x}\left(\log_e(ax)\right)^3$ **37.** $\frac{4\log_e(2x+3)}{2x+3}$ **38.** $\frac{1}{2}\left(\frac{x}{x-1}+\log_e(x-1)\right)$ **39.** $\frac{15x-8}{2\left(3x^2-2x\right)}$ **40.** $\frac{1}{2x\sqrt{4+\log_e x}}$ **41.** $\frac{1}{\sqrt{1+x^2}}$

3.3 Applications of differentiation

3.3.1 Tangents and normals

Example 35

Find the equation of the tangent to the curve $y = \ln(3x - 1)$ at the point $(2, \ln 5)$

Answer: $y = \frac{3}{5}x - \frac{6}{5} + \ln 5$

Example 36

(a) Show that the tangent of the curve to $y = \log_e x$ at $T(e, 1)$ has equation $x = ey$.

(b) Find the equation of the normal to $y = \log_e x$ at $T(e, 1)$.

(c) Sketch the curve, the tangent and the normal, and find the area of the triangle formed by the y axis and the tangent and normal at T.

Answer: (a) Show (b) $y = -ex + e^2 + 1$ (c) $\frac{1}{2}e\left(e^2 + 1\right)$

3.3.2 Curve sketching

Theorem 5

The function x .... over any logarithmic function $y = \log x$, as $x \to \infty$ or $x \to 0^+$.

$$\lim_{x\to\infty} \frac{\log x}{x} = \ldots\ldots \qquad \lim_{x\to 0^+} x \log x = \ldots$$

Timed exam practice 6 (Allow approximately 8 minutes)

[**2004 2U HSC Q9**] Consider the function $f(x) = \dfrac{\log_e x}{x}$ for $x > 0$.

i. Show that the graph of $y = f(x)$ has a stationary point at $x = e$. **2**

ii. By considering the gradient on either side of $x = e$, show that the stationary point is a maximum. **1**

iii. Use the fact that the maximum value of $f(x)$ occurs at $x = e$ to deduce that $e^x \geq x^e$ for all $x > 0$. **2**

Marking criteria

i. ✓ [1] Computes $f'(x)$ or equivalent progress
✓ [2] Correct solution

ii. ✓ [1] Correct solution

iii. ✓ [1] Observes that $\dfrac{\log_e x}{x} \leq \dfrac{1}{e}$ for all $x > 0$
✓ [2] Correct solution

Example 37

Consider the curve $f(x) = \log_e(4 - x^2)$.

(a) Find the value(s) of x for which $4 - x^2 > 0$.

(b) Hence write the domain of $f(x) = \log_e(4 - x^2)$.

(c) Find the x intercept(s) of $f(x) = \log_e(4 - x^2)$.

(d) Find the first and second derivatives of $f(x)$.

(e) Hence show that $f(x)$ has one stationary point and determine its nature.

(f) Sketch the curve $f(x) = \log_e(4 - x^2)$.

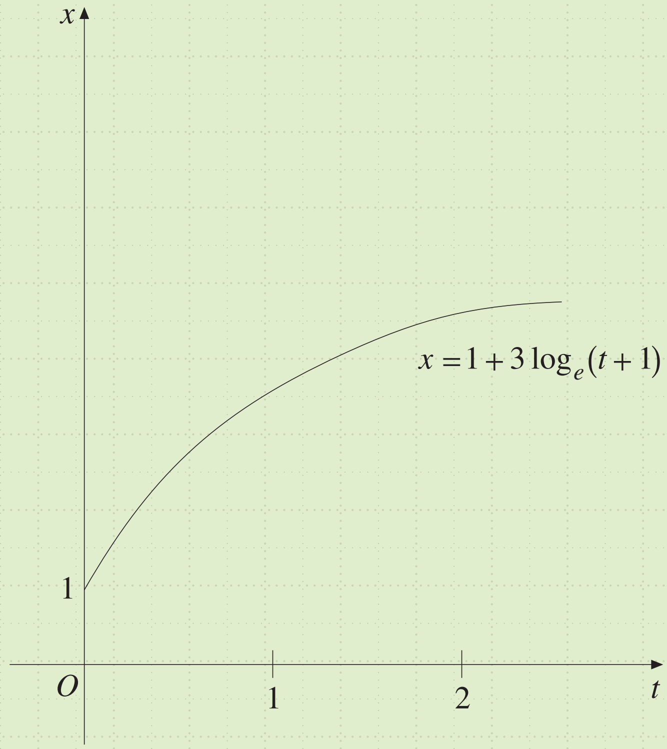
Answer: (a) $-2 < x < 2$ (b) As above (c) $x = \pm\sqrt{3}$ (d) $f'(x) = -\frac{2x}{4-x^2}$, $f''(x) = \frac{-2(4+x^2)}{(4-x^2)^2}$. (e) max: $(0, \ln 4)$

3.3.3 Optimisation/Motion

Example 38

[1996 2U HSC Q9] Two particles P and Q start moving along the x axis at time $t = 0$ and never meet. Particle P is initially at $x = 4$ and its velocity v at time t is given by $v = 2t + 4$.

The position of particle Q is given by $x = 1 + 3\log_e(t + 1)$. The diagram shows the graph of $x = 1 + 3\log_e(t + 1)$.



(a) Find an expression for the position of P at time t. **2**

(b) On the same set of axes, draw the graph of the function found in part (a). **2**

(c) P and Q are joined by an elastic string and M is the midpoint of the string. Show that the position of M at time t is given by **1**

$$x = \frac{1}{2}\left(t^2 + 4t + 3\log_e(t+1) + 5\right)$$

(d) Find the time at which the acceleration of M is zero. **3**

(e) Find the minimum distance between P and Q. **4**

Further exercises

Ⓐ **Ex 5H**

- Q4-15

(X1) **Ex 6H**

- Q4-15

3.3.4 Additional questions

1. Find the equation of the normal to the curve $y = \frac{1}{x} \ln x$ at $x = e$.

Section 4

Integration involving the logarithmic function

Learning Goal(s)

Knowledge
Various integration techniques involving the logarithmic function

Skills
Recognising the form $\frac{f'(x)}{f(x)}$ and finding primitives of these functions.

Understanding
Why $\int \frac{f'(x)}{f(x)}\,dx = \log_e(f(x)) + C$

By the end of this section am I able to:

22.13 Establish and use the formula $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + c$ and $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$ for $x \neq 0$, $f(x) \neq 0$ respectively

22.14 Establish and use the formulae $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + c$

22.15 Calculate the area under a curve

22.16 Calculate areas between curves determined by any functions within the scope of this syllabus

22.17 Use the Trapezoidal rule to estimate areas under curves

4.1 Primitives resulting in $\log_e x$

- Given $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$, then $\displaystyle\int \frac{1}{x}\,dx = \log_e |x| + C$

Laws/Results

- By the chain rule, $\dfrac{d}{dx}(\log_e f(x)) =$ ....
- Hence, $\displaystyle\int$ .... $dx = \log_e |f(x)| + C$.

4.1.1 Standard questions

Important note

When presented with a rational function as the integrand, look for the form ....

Example 39

Evaluate the following:

1. $\displaystyle\int \frac{dx}{2x+3}$.

2. $\displaystyle\int \frac{2}{3-9x}\,dx$

3. $\displaystyle\int \frac{3x}{1+6x^2}\,dx$

4. $\displaystyle\int \left(x + \frac{1}{x^2}\right)^2\,dx$

5. $\displaystyle\int \frac{e^x}{1+e^x}\,dx$

Answers

1. $\frac{1}{2}\log_e(2x+3)+C$ **2.** $-\frac{2}{9}\log_e(3-9x)+C$ **3.** $\frac{1}{4}\log_e(1+6x^2)+C$ **4.** $\frac{1}{3}x^3+2\ln x-\frac{1}{3}x^{-3}+C$ **5.** $\ln(1+e^x)+C$

Answers

1. $\frac{1}{2}\log_e(2x+3)+C$ **2.** $-\frac{2}{9}\log_e(3-9x)+C$ **3.** $\frac{1}{4}\log_e(1+6x^2)+C$ **4.** $\frac{1}{3}x^3+2\ln x-\frac{1}{3}x^{-3}+C$ **5.** $\ln(1+e^x)+C$

Example 40

Find the exact value of $\displaystyle\int_0^1 \frac{x}{x^2+3}\,dx$. **Answer:** $\frac{1}{2}\ln\frac{4}{3}$

4.1.2 Integrands requiring manipulation

Example 41

(a) Show that $\dfrac{x+7}{x-1}$ can be expressed as $1+\dfrac{8}{x-1}$.

(b) Hence evaluate $\displaystyle\int_2^{e+1} \frac{x+7}{x-1}\,dx$. **Answer:** $e+7$

Example 42

Show that $\dfrac{1}{x-1} - \dfrac{1}{x+1} = \dfrac{2}{x^2-1}$. Hence or otherwise, evaluate

$$\int_2^3 \frac{1}{x^2-1}\,dx$$

Answer: $\frac{1}{2}\ln\frac{3}{2}$

Example 43

(a) Differentiate $\dfrac{\log_e x}{x}$.

(b) Hence find the primitive of $\dfrac{\log_e x}{x^2}$.

Answer: (a) $\frac{1-\log_e x}{x^2}$ (b) $-\frac{1}{x} - \frac{\log_e x}{x} + C$

Example 44

Evaluate $\dfrac{d}{dx}\left(x^2 \log_e x\right)$ and hence evaluate $\displaystyle\int_1^e x \log_e x \, dx$.

Answer: $\frac{e^2+1}{4}$

Further exercises

(A) Ex 5I

- Q1-4, 6 last 2 columns
- Q5-18

(X1) Ex 6I

- Q2-8 last 2 columns
- Q9-16

4.2 Applications of integration

4.2.1 Simple areas

Example 45

Sketch the curve $y = 1 - \frac{3}{x}$. Hence or otherwise find the area bound by the curve $y = 1 - \frac{3}{x}$, the x axis and the line $x = 1$. **Answer:** $3\log_e 3 - 2$

Example 46

[2003 2U HSC] - modified Use the Trapezoidal Rule with three function values to find an approximation for

$$\int_2^6 \frac{x}{\log_e x}\,dx$$

Give your answer correct to 1 decimal place.

Example 47

(a) Sketch the curve $y = \dfrac{x+1}{x}$.

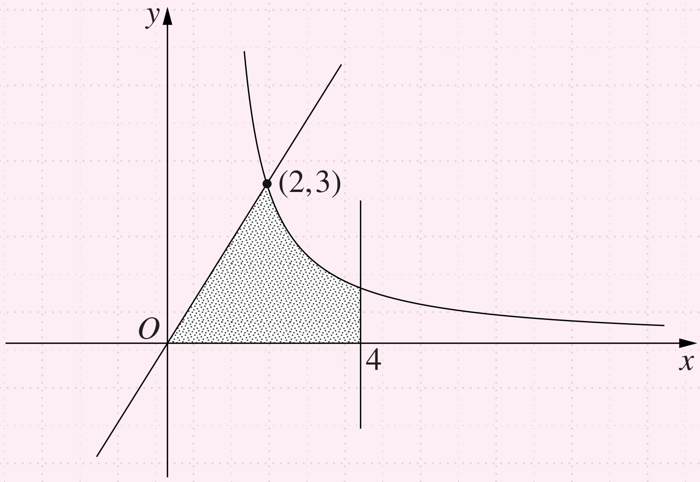
(b) Calculate the area enclosed by the curve, the x axis and the line $x = -5$.

Answer: $4 - \log_e 5$

Timed exam practice 7 (Allow approximately 5 minutes)

[2021 Adv HSC Q24] (3 marks) The curve $y = \dfrac{3}{x-1}$ intersects the line $y = \dfrac{3}{2}x$ at the point $(2, 3)$.

The region bounded by the curve $y = \dfrac{3}{x-1}$, the line $y = \dfrac{3}{2}x$, the x axis and the line $x = 4$ is shaded in the diagram.



Find the exact area of the shaded region.

Marking criteria

✓ [1] Calculates the area of the triangle, or equivalent merit

✓ [2] Evaluates $\displaystyle\int_2^4 \frac{3}{x-1}\,dx$, or equivalent merit

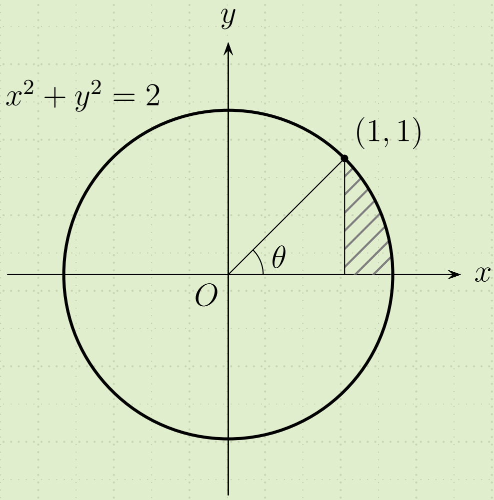
✓ [3] Provides the correct solution

4.2.2 Area between two curves

Example 48

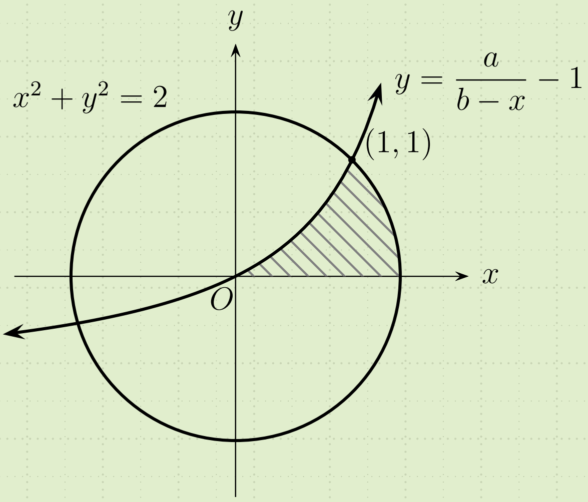
[2022 Adv HSC Q28] The graph of the circle $x^2 + y^2 = 2$ is shown.

The interval connecting the origin, O, and the point $(1, 1)$ makes an angle θ with the positive x axis.



(a) By considering the value of θ, find the exact area of the shaded region, as shown on the diagram. **2**

Part of the hyperbola $y = \dfrac{a}{b - x} - 1$ which passes through the points $(0, 0)$ and $(1, 1)$ is drawn with the circle $x^2 + y^2 = 2$ as shown.



(b) Show that $a = b = 2$. **2**

(c) Using parts (a) and (b), find the exact area of the region bounded by the hyperbola, the positive x axis and the circle as shown on the diagram. **3**

Answer: $2\ln 2 + \frac{\pi}{8} - \frac{3}{2}$

Example 49

[2009 2U HSC Q10] Let $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$.

(a) Show that the graph of $y = f(x)$ has no turning points. **2**

(b) Find the point of inflexion of $y = f(x)$. **1**

(c) i. Show that $1 - x + x^2 - \frac{1}{1+x} = \frac{x^3}{1+x}$ for $x \neq -1$. **1**

ii. Let $g(x) = \ln(1+x)$. **2**

Use the result in (c)i to show that $f'(x) \geq g'(x)$ for all $x \geq 0$.

(d) On the same set of axes, sketch the graphs of $y = f(x)$ and $y = g(x)$ for $x \geq 0$. **2**

(e) Show that $\frac{d}{dx}\left[(1+x)\ln(1+x) - (1+x)\right] = \ln(1+x)$. **2**

(f) Find the area enclosed by the graphs of $y = f(x)$ and $y = g(x)$, and the straight line $x = 1$. **2**

4.2.3 Inequalities

Example 50

[2002 Ext 1 HSC Q6] Let n be a positive integer.

i. By considering the graph of $y = \dfrac{1}{x}$ show that **2**

$$\frac{1}{n+1} < \int_n^{n+1} \frac{dx}{x} < \frac{1}{n}$$

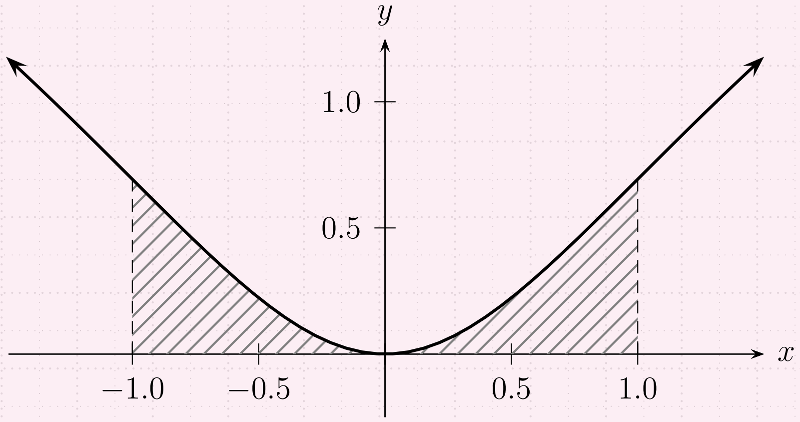
ii. Hence deduce that **3**

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

4.2.4 Trapezoidal Rule

Timed exam practice 8 (Allow approximately 10 minutes)

[2024 Adv HSC Q22] (6 marks) The graph of the function $f(x) = \ln(1+x^2)$ is shown.



(a) Prove that $f(x)$ is concave up for $-1 < x < 1$. **3**

(b) A table of function values, correct to 4 decimal places, for some x values is provided. **2**

x	0	0.25	0.5	0.75	1
$\ln(1+x^2)$	0	0.0606	0.2231	0.4463	0.6931

Using the function values provided and the trapezoidal rule, estimate the shaded area in the diagram.

(c) Is the answer to part (b) an overestimate or underestimate? Give a reason for your answer. **1**

Important note

⚠ The word *hence* may not appear explicitly!

Marking criteria

(a) ✓ [1] Finds the correct first derivative, or equivalent merit
✓ [2] Finds the correct second derivative, or equivalent merit
✓ [3] Provides correct solution

(b) ✓ [1] Uses the values from the table correctly in the trapezoidal rule, or equivalent merit
✓ [2] Provides correct solution

(c) ✓ [1] Provides correct answer with a reason

4.2.5 Motion

Example 51

[2000 2U HSC Q8] A particle is moving in a straight line, starting from the origin. At time t seconds the particle has a displacement of x metres from the origin and a velocity v ms^{-1}. The displacement is given by $x = 2t - 3\log_e(t+1)$.

(i) Find an expression for v **1**

(ii) Find the initial velocity. **1**

(iii) Find when the particle comes to rest. **2**

(iv) Find the distance travelled by the particle in the first three seconds. **3**

Answer: (i) $\dot{x} = 2 - \frac{3}{t+1}$ (ii) -1 (iii) $t = \frac{1}{2}$ (iv) $4 + 3\log_e \frac{9}{16}$ m

Example 52

[2007 2U HSC Q5] A particle is moving on the x axis and is initially at the origin. Its velocity, v metres per second, at time t seconds is given by

$$v = \frac{2t}{16 + t^2}$$

i. What is the initial velocity of the particle? **1**

ii. Find an expression for the acceleration of the particle. **2**

iii. Find the time when the acceleration of the particle is zero. **1**

iv. Find the position of the particle when $t = 4$. **3**

Example 53

[2000 2U HSC Q10] ⚠ The first snow of the season begins to fall during the night. The depth of the snow, h, increases at a constant rate through the night and the following day. At 6 am a snow plough begins to clear the road of snow. The speed, v km/h, of the snow plough is inversely proportional to the depth of snow. (This means $v = \frac{A}{h}$ where A is a constant)

Let x km be the distance the snow plough has cleared and let t be the time in hours from the beginning of the snowfall. Let $t = T$ correspond to 6 am.

i. Explain carefully why, for $t \geq T$, **3**

$$\frac{dx}{dt} = \frac{k}{t}$$

where k is a constant.

ii. In the period from 6 am to 8 am the snow plough clears 1 km of road, but it takes a further 3.5 hours to clear the next kilometre. **3**

At what time did it begin snowing? **Answer:** 3:20 am

Further exercises

Applications of integration

Ⓐ **Ex 5J**

- Q4-18

Ⓐ **Ex 7C**

- Q8(h), Q15

Ⓐ **Ex 7E**

- Q6

(x1) **Ex 6J**

- Q2-18

(x1) **Ex 9C**

- Q4(a)i, 11, 14

(x1) **Ex 9F**

- Q3

Calculus with other bases

Ⓐ **Ex 5K**

- Q6-16

(x1) **Ex 6K**

- Q6-18

Watch: http://youtu.be/UFgod5tmLYY

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

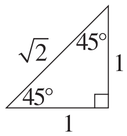
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$



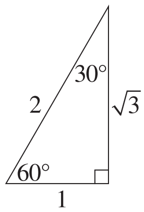
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

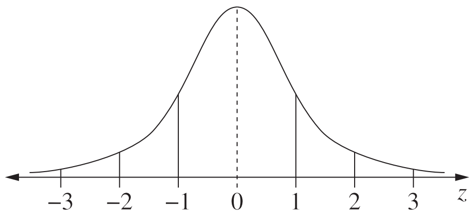
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n - r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n - x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^{n} = x^{n} + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^{r} + \cdots + a^{n}$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^{2}+y^{2}}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_{1}x_{2} + y_{1}y_{2},$$

where $\underset{\sim}{u} = x_{1}\underset{\sim}{i} + y_{1}\underset{\sim}{j}$

and $\underset{\sim}{v} = x_{2}\underset{\sim}{i} + y_{2}\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^{n} = r^{n}(\cos n\theta + i\sin n\theta)$$
$$= r^{n}e^{in\theta}$$

Mechanics

$$\frac{d^{2}x}{dt^{2}} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^{2}(x-c)$$

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